



## Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact [support@jstor.org](mailto:support@jstor.org).

# THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XIII.

MARCH, 1906.

No. 3.

## CERTAIN FACTORS OF THE GROUP DETERMINANT.

By WILLIAM BENJAMIN FITE.

So far as I know the following method for finding factors of a group determinant is new.

To the  $n$  operators  $s_1, s_2, \dots, s_n$  of a group  $G$  of finite order we make correspond  $n$  independent variables  $x_{s_1}, x_{s_2}, \dots, x_{s_n}$ , such that  $x_{s_k} \equiv x_{s_i s_j}$  if  $s_k = s_i s_j$ . With these variables we form the following determinant, which is called the group determinant of  $G$ :

$$\Theta(x) \equiv \begin{vmatrix} x_{s_1 s_1^{-1}} & x_{s_1 s_2^{-1}} & \dots & x_{s_1 s_n^{-1}} \\ x_{s_2 s_1^{-1}} & x_{s_2 s_2^{-1}} & \dots & x_{s_2 s_n^{-1}} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ x_{s_n s_1^{-1}} & x_{s_n s_2^{-1}} & \dots & x_{s_n s_n^{-1}} \end{vmatrix}$$

If we take for  $G$  the symmetric group of degree 3 and put  $s_1 \equiv 1, s_2 \equiv (abc), s_3 \equiv (acb), s_4 \equiv (ab), s_5 \equiv (bc), s_6 \equiv (ac)$ , then

$$\Theta(x) \equiv \begin{vmatrix} x_{s_1} & x_{s_2} & x_{s_3} & x_{s_4} & x_{s_5} & x_{s_6} \\ x_{s_2} & x_{s_1} & x_{s_3} & x_{s_6} & x_{s_5} & x_{s_4} \\ x_{s_3} & x_{s_2} & x_{s_1} & x_{s_6} & x_{s_4} & x_{s_5} \\ x_{s_4} & x_{s_5} & x_{s_6} & x_{s_1} & x_{s_3} & x_{s_2} \\ x_{s_5} & x_{s_6} & x_{s_4} & x_{s_2} & x_{s_1} & x_{s_3} \\ x_{s_6} & x_{s_4} & x_{s_5} & x_{s_3} & x_{s_2} & x_{s_1} \end{vmatrix}$$

Let  $H$ , of order  $h$ , be any subgroup of  $G$ , and arrange the elements of the first column of  $\Theta(x)$  with respect to the divisions  $s_i H$ . Then  $\Theta(x)$  can be divided up into squares of  $h$  elements each, such that in any square the sums of the elements of the  $h$  rows are the same. For, the subscripts of the elements in the square containing the  $i$ th row and the  $j$ th column are contained in  $s_i H \cdot H s_{j-1}$ , the subscripts for the different rows being obtained by taking the different operators in the first  $H$ , and those for the different columns by taking the different operators in the second  $H$ .

We now form the determinant  $\Theta_1$ , of degree  $n/h$ , by taking for each element the sum of the elements in any row of the corresponding square of  $\Theta(x)$ . If we arrange the elements of the determinant of the symmetric group of degree 3 with respect to the subgroup  $[1, (ab)]$ , the corresponding  $\Theta_1$  is

$$\begin{vmatrix} x_{s_1} + x_{s_4} & x_{s_3} + x_{s_5} & x_{s_2} + x_{s_6} \\ x_{s_2} + x_{s_5} & x_{s_1} + x_{s_6} & x_{s_3} + x_{s_4} \\ x_{s_3} + x_{s_6} & x_{s_2} + x_{s_4} & x_{s_1} + x_{s_5} \end{vmatrix}$$

For convenience I shall hereafter omit the  $x$ 's and write only the subscripts.

I wish to prove that  $\Theta_1$  is a factor of  $\Theta$ . (a) Add the elements of the first  $h$  rows of  $\Theta$  to the corresponding elements of the first row; the elements of the second  $h$  rows to the corresponding elements of the  $(h+1)$ th row; and so on. (b) Now so re-arrange the rows and columns that the minor formed by the first  $h$  rows and columns is  $\Theta_1$ . (c) Then multiply the elements of every row, except the first  $h$  rows, by  $h$ . (d) After this, subtract the elements of the first row from the corresponding elements of each row that was originally among the first  $h$  rows; the elements of the second row from the corresponding elements of each row that was originally among the second  $h$  rows; and so on. (e) Now to the elements of the first column add the corresponding elements of the columns that were originally the first  $h$  columns; to the elements of the second column add the corresponding elements of the columns that were originally among the second  $h$  columns; and so on. In the final form of the determinant each element of the first diagonal square of order  $h$  is  $h$  times the corresponding element of  $\Theta_1$  and all the other elements of the first  $h$  columns are zero. Moreover, a further obvious modification shows that the remaining elements of the first  $h$  rows can all be made zero. Therefore  $\Theta_1$  is a factor of  $\Theta$ .

I shall illustrate the steps of this proof by the determinant of the symmetric group of degree 3.

$$\Theta \equiv \begin{vmatrix} s_1 & s_4 & s_3 & s_5 & s_2 & s_6 \\ s_4 & s_1 & s_5 & s_3 & s_6 & s_2 \\ s_2 & s_5 & s_1 & s_6 & s_3 & s_4 \\ s_5 & s_2 & s_6 & s_1 & s_4 & s_3 \\ s_3 & s_6 & s_2 & s_4 & s_1 & s_5 \\ s_6 & s_3 & s_4 & s_2 & s_5 & s_1 \end{vmatrix}$$

$$\begin{aligned}
(a) \quad & \begin{vmatrix} s_1+s_4 & s_4+s_1 & s_3+s_5 & s_5+s_3 & s_2+s_6 & s_6+s_2 \\ s_4 & s_1 & s_6 & s_3 & s_6 & s_2 \\ s_2+s_5 & s_5+s_2 & s_1+s_6 & s_6+s_1 & s_3+s_4 & s_4+s_3 \\ s_5 & s_2 & s_6 & s_1 & s_4 & s_3 \\ s_3+s_6 & s_6+s_3 & s_2+s_4 & s_4+s_2 & s_1+s_5 & s_5+s_1 \\ s_6 & s_3 & s_4 & s_2 & s_5 & s_1 \end{vmatrix} \\
(b), (c) \quad & \begin{vmatrix} s_1+s_4 & s_3+s_5 & s_2+s_6 & s_4+s_1 & s_5+s_3 & s_6+s_2 \\ s_2+s_5 & s_1+s_6 & s_3+s_4 & s_5+s_2 & s_6+s_1 & s_4+s_3 \\ s_3+s_6 & s_2+s_4 & s_1+s_5 & s_6+s_3 & s_4+s_2 & s_5+s_1 \\ 2s_4 & 2s_5 & 2s_6 & 2s_1 & 2s_3 & 2s_2 \\ 2s_5 & 2s_6 & 2s_4 & 2s_2 & 2s_1 & 2s_3 \\ 2s_6 & 2s_4 & 2s_5 & 2s_3 & 2s_2 & 2s_1 \end{vmatrix} \\
(d) \quad & \begin{vmatrix} s_1+s_4 & s_3+s_5 & s_2+s_6 & s_4+s_1 & s_5+s_3 & s_6+s_2 \\ s_2+s_5 & s_1+s_6 & s_3+s_4 & s_5+s_2 & s_6+s_1 & s_4+s_3 \\ s_3+s_6 & s_2+s_4 & s_1+s_5 & s_6+s_3 & s_4+s_2 & s_5+s_1 \\ s_4-s_1 & s_5-s_3 & s_6-s_2 & s_1-s_4 & s_3-s_5 & s_2-s_6 \\ s_5-s_2 & s_6-s_1 & s_4-s_3 & s_2-s_5 & s_1-s_6 & s_3-s_4 \\ s_6-s_3 & s_4-s_2 & s_5-s_1 & s_3-s_6 & s_2-s_4 & s_1-s_5 \end{vmatrix} \\
(e) \quad & \begin{vmatrix} 2(s_1+s_4) & 2(s_3+s_5) & 2(s_2+s_6) & s_4+s_1 & s_5+s_3 & s_6+s_2 \\ 2(s_2+s_5) & 2(s_1+s_6) & 2(s_3+s_4) & s_5+s_2 & s_6+s_1 & s_4+s_3 \\ 2(s_3+s_6) & 2(s_2+s_4) & 2(s_1+s_5) & s_6+s_3 & s_4+s_2 & s_5+s_1 \\ 0 & 0 & 0 & s_1-s_4 & s_3-s_5 & s_2-s_6 \\ 0 & 0 & 0 & s_2-s_5 & s_1-s_6 & s_3-s_4 \\ 0 & 0 & 0 & s_3-s_6 & s_2-s_4 & s_1-s_5 \end{vmatrix}
\end{aligned}$$

If  $H'$  is conjugate to  $H$  in  $G$ , the corresponding factor  $\Theta'_1$  is the same as  $\Theta_1$ . For, the element in the  $i$ th row and  $j$ th column of  $\Theta_1$  is  $\Sigma s_a$ , where  $s_a$  runs through the operators of  $s_i H s_j^{-1}$ . If  $s_r H s_r^{-1} = H'$ , the element in the  $i$ th row and  $j$ th column of  $\Theta_1$  is the same as the element in the  $k$ th row and  $l$ th column of  $\Theta'_1$ , where  $s_i = s_k s_r$ ,  $s_j^{-1} = s_r^{-1} s_l^{-1}$ , since  $s_i H s_j^{-1} = s_k s_r H s_r^{-1} s_l^{-1} = s_k H' s_l^{-1}$ . Moreover,  $s_j$  fixes  $s_l$ , and therefore the elements of the  $j$ th column of  $\Theta_1$  are the same, except as to arrangement, as those of the  $l$ th column of  $\Theta'_1$ . Likewise the elements of the  $i$ th row of  $\Theta_1$  are the same, except as to arrangement, as those of the  $k$ th row of  $\Theta'_1$ . Therefore  $\Theta'_1 = \pm \Theta_1$ .